

Empirical process of residuals for regression models with long memory errors

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Abstract

We consider the residual empirical process in random design regression with long memory errors. We establish its limiting behaviour, showing that its rates of convergence are different from the rates of convergence for to the empirical process based on (unobserved) errors. Also, we study a residual empirical process with estimated parameters. Its asymptotic distribution can be used to construct Kolmogorov-Smirnov, Cramér-Smirnov-von Mises, or other goodness-of-fit tests. Theoretical results are justified by simulation studies.

1 Introduction

Consider a random design regression model,

$$Y_i = m(X_i) + \varepsilon_i, \quad i = 1, \dots, n, \quad (1)$$

where $\{X, X_i, i \geq 1\}$ is a stationary sequence of random variables with a density $f = f_X$, independent of a centered, stationary long memory error sequence $\{\varepsilon, \varepsilon_i, -\infty < i < \infty\}$, with a distribution F_ε and density f_ε . The goal of this paper is to study the asymptotic properties of the empirical process of residuals,

$$\hat{K}_n(x) := \sum_{i=1}^n (\mathbf{1}_{\{\hat{\varepsilon}_i \leq x\}} - F_\varepsilon(x)),$$

where $\hat{\varepsilon}_i = Y_i - \hat{m}(X_i)$ and $\hat{m}(\cdot)$ is an estimator of the function $m(\cdot)$.

Residual-based inference is a standard tool in regression analysis. With this in mind, several authors considered empirical process of residuals in case of independent random variables or weakly dependent stationary time series, see e.g. [3], [10], [1], [17], [5], [18], just to mention few.

As for regression models with long memory errors, in [4], the authors obtained that in case of a parametric regression, $m(x) = \beta_0 + \beta_1 x$, with *a known*

intercept, the limiting behaviour of $\hat{K}_n(\cdot)$ is similar to the limiting behaviour of

$$K_n(x) := \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq x\}} - F_\varepsilon(x)),$$

in the sense that $\sigma_{n,1}^{-1}K_n(\cdot)$ and $\sigma_{n,1}^{-1}\hat{K}_n(\cdot)$ converge weakly to, respectively, $f_\varepsilon(x)Z_1$, where Z_1 is standard normal and $\sigma_{n,1}$ is an appropriate scaling factor. However, if one considers a parametric regression when both slope and intercept are unknown, from the latter paper one can only conclude that

$$\sigma_{n,1}^{-1} \sup_{x \in \mathbb{R}} |\hat{K}_n(x)| = o_P(1).$$

To see intuitively why this is the case, consider for a moment a simple model $Y_i = \mu + \varepsilon_i$. Estimate μ by the sample mean \bar{Y} and compute residuals $\hat{\varepsilon}_i = Y_i - \bar{Y}$. Then

$$\hat{K}_n(x) = K_n(x + \bar{Y}) + (F_\varepsilon(x + \bar{Y}) - F_\varepsilon(x)).$$

From a general theory for empirical processes based on long memory random variables ε_i , $i \geq 1$, we conclude

$$\sigma_{n,1}^{-1} \sup_x \left| K_n(x) + f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i \right| = o_P(1),$$

see e.g. [6], [8], [19]. On the other hand, using the Taylor's expansion,

$$(F_\varepsilon(x + \bar{Y}) - F_\varepsilon(x)) \approx f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i.$$

Therefore, the contribution of $\bar{\varepsilon} = \sum_{i=1}^n \varepsilon_i$ cancels out and asymptotic behaviour of $\hat{K}_n(x)$ cannot be concluded from that of $K_n(x)$. See [13] and [12] for precise results along these lines.

The main goal of this paper is to establish a general theory on asymptotic behaviour for $\hat{K}_n(\cdot)$. In particular, this theory is applied to the parametric regression and a nonparametric regression; the latter in a longer, arxiv version of the paper. We will show in this paper, that convergence properties of $\hat{K}_n(\cdot)$ may be completely different from the asymptotics of $K_n(\cdot)$. To do this, we will establish a second order expansion for $\hat{K}_n(\cdot)$ (see Theorems 3.1 and 3.2).

The established results can be used, in principle, to test whether the errors $\varepsilon_1, \dots, \varepsilon_n$ are consistent with a given distribution F_ε . If F_ε belongs to a one-parameter family $\{F_\varepsilon(\cdot, \theta), \theta \in \mathbb{R}\}$, then one needs to know the value of the parameter θ . Therefore, we discuss asymptotic properties of an empirical process of residuals, when a parameter θ is estimated. The appropriate limit theorems are established in Section 3.4. Our theoretical results are confirmed by small simulation studies in Section 4.

The results for empirical processes in Sections 3.3 can be applied directly to establish limiting behaviour of quantiles (see [8, Section 5]). Furthermore, in a spirit of [8, Section 3], our results should be applicable to the error density estimation. However, a precise proof requires at least third order expansion of the residual-based empirical process (see Section 3.6). Finally, it would be interesting to establish corresponding results in case of fixed-design regression.

2 Preliminaries: LRD error sequence

In the sequel, $F_U(\cdot)$, $f_U(\cdot)$ denote a distribution and a density, respectively, of a given random variable U . Also, if U has finite mean, we denote $U^* = U - \mathbb{E}[U]$.

We shall consider the following assumption on the error sequence:

(E) ε_i , $i \geq 1$, is an infinite order moving average

$$\varepsilon_i = \sum_{k=0}^{\infty} c_k \eta_{i-k}, \quad \text{with } c_0 = 1,$$

where η_i , $-\infty < i < \infty$, is a sequence of centered i.i.d. random variables, independent of X_i , $i \geq 1$. We assume that $\mathbb{E}[\varepsilon^4] < \infty$, $\mathbb{E}[\varepsilon_1^2] = 1$, and for some $\alpha_\varepsilon \in (0, 1)$, $c_k \sim k^{-(\alpha_\varepsilon+1)/2} L_0(k)$ as $k \rightarrow \infty$, where $L_0(\cdot)$ is slowly varying at infinity.

Let

$$\varepsilon_{n,r} = \sum_{i=1}^n \sum_{1 \leq j_1 < \dots < j_r=1}^n \prod_{s=1}^r c_{j_s} \eta_{i-j_s}. \quad (2)$$

In particular, $\varepsilon_{n,1} = \sum_{i=1}^n \varepsilon_i$ and if $r\alpha_\varepsilon < 1$,

$$\sigma_{n,r}^2 := \text{Var}(\varepsilon_{n,r}) \sim n^{2-r\alpha_\varepsilon} L_0^{2r}(n). \quad (3)$$

From [8] we know that for $r < \alpha_\varepsilon^{-1}$, as $n \rightarrow \infty$,

$$\sigma_{n,r}^{-1} \varepsilon_{n,r} \xrightarrow{d} Z_r, \quad r = 1, 2, \quad (4)$$

where Z_r is a random variable which can be represented by a multiple Wiener-Itô integral. In particular, Z_1 is standard normal. Moreover, the random variables Z_1, \dots, Z_p are uncorrelated, see e.g. [11, Eq. (1.22)]. We also note that the convergence in (4) holds jointly.

Furthermore, let

$$S_{n,p}(x) = \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq x\}} - F_\varepsilon(x)) + \sum_{r=1}^p (-1)^{r-1} F_\varepsilon^{(r)}(x) \varepsilon_{n,r}.$$

Assume that $F_\eta(\cdot)$ is 5 times differentiable with bounded, continuous and integrable derivatives. We note in passing that these properties are transferable to

$F_\varepsilon(\cdot)$. Following [19, Theorem 3] and [8, Theorem 2.2] we conclude, in particular, that for $\alpha_\varepsilon < 1/2$,

$$\sigma_{n,2}^{-1} S_{n,1}(x) \Rightarrow f_\varepsilon^{(1)}(x) Z_2, \quad (5)$$

where Z_2 is the same random variable as in (4). Otherwise, if $\alpha_\varepsilon > 1/2$, then

$$n^{-1/2} S_{n,1}(x) \Rightarrow W_1(x), \quad (6)$$

where $\{W_1(x), x \in \mathbb{R}\}$ is a Gaussian process and \Rightarrow denotes weak convergence in $D[0, 1]$. Furthermore, for $\alpha_\varepsilon > 1/3$,

$$\sigma_{n,2}^{-1} \sup_{x \in \mathbb{R}} |S_{n,2}(x)| \xrightarrow{a.s.} 0. \quad (7)$$

The structure of this Gaussian process and its covariance is given in a rather complicated form; see [19] for more details.

3 Results

Let

$$\Delta_i := \varepsilon_i - \hat{\varepsilon}_i = \varepsilon_i - (Y_i - \hat{m}(X_i)) = \hat{m}(X_i) - m(X_i), \quad \Delta = (\Delta_1, \dots, \Delta_n).$$

3.1 Empirical process of residuals: $\alpha_\varepsilon < 1/2$

The following result provides an uniform expansion of the process $\hat{K}_n(\cdot)$ and forms a basis for further analysis.

Theorem 3.1. *Assume (E) with $\alpha_\varepsilon < 1/2$. Assume that $F_\eta(\cdot)$ is 3 times differentiable with bounded, continuous and integrable derivatives. Suppose that Δ can be written as $\Delta_0 \mathbf{1} + (\Delta_{01}, \dots, \Delta_{0n})$, where*

$$\delta_n + \frac{n^2 \delta_n}{\sigma_{n,2}^4} + \frac{n^2 \delta_n^2}{\sigma_{n,2}^3} + \frac{n \delta_n}{\sigma_{n,2}^3} \rightarrow 0; \quad (8)$$

- $\frac{1}{\delta_n} \Delta_0 \xrightarrow{d} V$, where V is a nondegenerate random variable;
- $|\Delta_{0i}| = o_P(\delta_n)$, uniformly in i .

Then

$$\begin{aligned} & \sup_{x \in \mathbb{R}} \left| \hat{K}_n(x) - K_n(x) - f_\varepsilon(x) \sum_{i=1}^n \Delta_i - \frac{1}{2} f_\varepsilon^{(1)}(x) \sum_{i=1}^n \Delta_i^2 + f_\varepsilon^{(1)}(x) \Delta_0 \varepsilon_{n,1} \right| \\ &= O_P(\delta_n^{1-\nu} \sigma_{n,2}) + o_P(\delta_n \sigma_{n,1}) + O_P\left(\sum_{i=1}^n \Delta_i^3\right). \end{aligned}$$

In principle, this result is very similar to [4, Theorem 2.1]. However, we provide $o_P(\cdot)$ rates of the approximation. This is crucial to establish limit theorems for the process $\hat{K}_n(\cdot)$.

To have some intuition, let us write

$$\begin{aligned}\hat{K}_n(x) - K_n(x) &= \sum_{i=1}^n \mathbf{1}_{\{\varepsilon_i \leq x + \Delta_i\}} - \sum_{i=1}^n \mathbf{1}_{\{\varepsilon_i \leq x\}} \\ &= \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq x + \Delta_i\}} - F_\varepsilon(x + \Delta_i)) - \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq x\}} - F_\varepsilon(x)) \\ &\quad + \sum_{i=1}^n (F_\varepsilon(x + \Delta_i) - F_\varepsilon(x)).\end{aligned}\tag{9}$$

From Theorem 3.1 and (7) we conclude for $\alpha_\varepsilon < 1/2$ that, uniformly in x ,

$$\begin{aligned}\hat{K}_n(x) &= K_n(x) + f_\varepsilon(x) \sum_{i=1}^n \Delta_i + \frac{1}{2} f'_\varepsilon(x) \sum_{i=1}^n \Delta_i^2 - f_\varepsilon^{(1)}(x) \Delta_0 \varepsilon_{n,1} \\ &\quad + o_P(\sigma_{n,2} + \delta_n \sigma_{n,1}) + O_P\left(\sum_{i=1}^n \Delta_i^3\right) \\ &= -f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i + f_\varepsilon(x) \sum_{i=1}^n \Delta_i + f_\varepsilon^{(1)}(x) \varepsilon_{n,2} + \frac{1}{2} f_\varepsilon^{(1)}(x) \sum_{i=1}^n \Delta_i^2 - f_\varepsilon^{(1)}(x) \Delta_0 \varepsilon_{n,1} \\ &\quad + o_P(\sigma_{n,2} + \delta_n \sigma_{n,1}) + O_P\left(\sum_{i=1}^n \Delta_i^3\right).\end{aligned}\tag{10}$$

We note in passing that in order to obtain the above expansion via (7) one has to assume that $F_\eta(\cdot)$ is 5 times differentiable.

As we will see below (Section 3.3), it may happen that the first order contribution

$$-f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i + f_\varepsilon(x) \sum_{i=1}^n \Delta_i$$

is negligible. In other words, the rates of convergence of $\hat{K}_n(\cdot)$ will be different from those for $K_n(\cdot)$. The rates of convergence will be determined by the second order term

$$f_\varepsilon^{(1)}(x) \varepsilon_{n,2} + \frac{1}{2} f_\varepsilon^{(1)}(x) \sum_{i=1}^n \Delta_i^2 - f_\varepsilon^{(1)}(x) \Delta_0 \varepsilon_{n,1}.$$

3.2 Empirical process of residuals: $\alpha_\varepsilon > 1/2$

Let $\xi_i = \varepsilon_i - \eta_i$. Define $\xi_{n,r}$ in the analogous way as $\varepsilon_{n,r}$; see (2).

Theorem 3.2. Assume (E) with $\alpha_\varepsilon > 1/2$. Assume that $F_\eta(\cdot)$ is 3 times differentiable with bounded, continuous and integrable derivatives. Suppose that Δ can be written as $\Delta_0 \mathbf{1} + (\Delta_{01}, \dots, \Delta_{0n})$, where

$$\delta_n + \sqrt{n}\delta_n^2 \rightarrow 0; \quad (11)$$

- $\frac{1}{\delta_n} \Delta_0 \xrightarrow{d} V$, where V is a nondegenerate random variable;
- $|\Delta_{0i}| = o_P(\delta_n)$, uniformly in i .

Then

$$\begin{aligned} \hat{K}_n(x) &= K_n(x) + f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i + \left(f_\varepsilon(x) \sum_{i=1}^n \Delta_i - f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i - f_\varepsilon^{(1)}(x) \Delta_0 \xi_{n,1} \right) \\ &\quad + O\left(\sum_{i=1}^n \Delta_i^2 \right) + o_P(\sqrt{n}) + O_P(\delta_n \sigma_{n,1}) \end{aligned}$$

where $n^{-1/2} (K_n(x) + f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i)$ converges weakly to $W_1(x)$ from (6).

3.3 Application to parametric regression

The results of Theorems 3.1 and 3.2 are the tools to establish a limit theorem for $\hat{K}_n(\cdot)$ in case of parametric model

$$m(x) = \beta_0 + \beta_1 x. \quad (12)$$

We assume that the regression parameters are estimated using standard least squares. We make the following assumption on the predictors X_i , $i \geq 1$:

(P) X_i , $i \geq 1$, is a random sequence such that $\sup_i E[|X_i| + |\bar{X}|] < \infty$.

Corollary 3.3. Assume (P) and (E) and that

$$\hat{\beta}_1 - \beta_1 = o_P(\sigma_{n,1}/n). \quad (13)$$

Assume that $F_\eta(\cdot)$ is 5 times differentiable with bounded, continuous and integrable derivatives.

(a) If $\alpha_\varepsilon < 1/2$, then

$$\frac{1}{\sigma_{n,2}} \hat{K}_n(x) \Rightarrow f_\varepsilon^{(1)}(x) (Z_2 - \frac{1}{2} Z_1^2),$$

where Z_1, Z_2 are defined in (4).

(b) If $\alpha_\varepsilon > 1/2$, then $n^{-1/2} \hat{K}_n(x) \Rightarrow W_1(x)$.

Remark 3.4. Note that the rate of convergence $\sigma_{n,1}$ for the original process $K_n(\cdot)$ changes to $\sigma_{n,2}$ or \sqrt{n} for $\hat{K}_n(\cdot)$. The similar phenomena was observed in a context of empirical processes with estimated parameters in [13] (see also [2]). Note further that a possible LRD of predictors does not play any role.

Furthermore, from the proof of Corollary 3.3 below, we may conclude that in case $\beta_0 = 0$ the limiting behaviour of $K_n(x)$ and $\hat{K}_n(x)$ is the same. In other words, for the model (12) with $\beta_0 = 0$, we have (see also [4])

$$\sigma_{n,1}^{-1} \hat{K}_n(x) \Rightarrow f_\varepsilon(x) Z_1.$$

Remark 3.5. The condition (13) can be verified for many stationary sequences. In particular, if X_i , $i \geq 1$, is LRD linear sequence with parameter α_X , then the rate of convergence of $(\hat{\beta}_1 - \beta_1)$ is either \sqrt{n} or $n^{(\alpha_X + \alpha_\varepsilon)/2}$, for $\alpha_X + \alpha_\varepsilon > 1$ or $\max(\alpha_X, \alpha_\varepsilon) < 1/2$, respectively; see [16] and [7].

Proof of Corollary 3.3. Least squares estimation leads to the following expressions:

$$\hat{\beta}_1 - \beta_1 = \frac{1}{s_n} \left(\frac{1}{n} \sum_{j=1}^n X_j \varepsilon_j - \bar{X} \bar{\varepsilon} \right), \quad \hat{\beta}_0 - \beta_0 = \bar{\varepsilon} - \bar{X}(\hat{\beta}_1 - \beta_1), \quad (14)$$

where \bar{X} and $\bar{\varepsilon}$ are sample means based on X_1, \dots, X_n and $\varepsilon_1, \dots, \varepsilon_n$, respectively, and $s_n = \frac{1}{n} \sum_{j=1}^n (X_j - \bar{X})^2$. We have

$$\Delta_i = \hat{m}(X_i) - m(X_i) = (\hat{\beta}_0 - \beta_0) + (\hat{\beta}_1 - \beta_1)X_i = \bar{\varepsilon} + (\hat{\beta}_1 - \beta_1)(X_i - \bar{X}). \quad (15)$$

From (3) we conclude that

$$\bar{\varepsilon} = O_P(\sigma_{n,1}/n), \quad \sigma_{n,1}^2/n \sim \sigma_{n,2}, \quad \text{as } n \rightarrow \infty. \quad (16)$$

From (13) and Assumption (P) we conclude $\Delta_i = \bar{\varepsilon} + o_P(\sigma_{n,1}/n)O_P(1)$. Let now $\delta_n = \sigma_{n,1}/n$. It is straightforward to check that such δ_n fulfills (8). Therefore, the conditions of Theorem 3.1 are fulfilled with $\Delta_0 = \bar{\varepsilon}$ and $V = Z_1$.

Furthermore, from (15), $\sum_{i=1}^n \Delta_i = n\bar{\varepsilon} = \varepsilon_{n,1} = \sum_{i=1}^n \varepsilon_i$ and via (16),

$$\sum_{i=1}^n \Delta_i^2 = n\bar{\varepsilon}^2 + n\bar{\varepsilon}O_P(\sigma_{n,1}/n) + o_P(n\sigma_{n,1}^2/n^2) = n\bar{\varepsilon}^2 + o_P(\sigma_{n,2}). \quad (17)$$

Consequently, noting that $\delta_n \sigma_{n,1} \sim \sigma_{n,2}$ and $n\bar{\varepsilon}^2 = \bar{\varepsilon}\varepsilon_{n,1}$, the expansion (10) reads

$$\hat{K}_n(x) = f_\varepsilon^{(1)}(x)\varepsilon_{n,2} - \frac{1}{2}f_\varepsilon^{(1)}(x)n\bar{\varepsilon}^2 + o_P(\sigma_{n,2}) =: S_n(x) + o_P(\sigma_{n,2}), \quad (18)$$

uniformly in x . The result of part (a) follows now from (4).

As for part (b), we recall that $\sum_{i=1}^n \Delta_i - \sum_{i=1}^n \varepsilon_i = 0$. Also, since $\alpha_\varepsilon > 1/2$, $\Delta_0 \xi_{n,1} = O_P(\sigma_{n,1}^2/n) = O_P(\sigma_{n,2})$ and via (17), $\sum_{i=1}^n \Delta_i^2 = O_P(\sigma_{n,2}) = o_P(\sqrt{n})$. Finally, the choice of δ_n yields $\delta_n \sigma_{n,1} = o_P(\sqrt{n})$. Therefore, part (b) follows from Theorem 3.2. \square

3.4 Residual empirical process with estimated parameters

Let us focus on the parametric regression model of Section 3.3. From Corollary 3.3

$$\frac{1}{\sigma_{n,2}} \sup_{x \in \mathbb{R}} |\hat{K}_n(x)| \xrightarrow{d} \sup_{x \in \mathbb{R}} |f_\varepsilon^{(1)}(x)| (Z_2 - \frac{1}{2} Z_1^2),$$

for $\alpha_\varepsilon < 1/2$. The above result can be used, in principle, to test whether the errors $\varepsilon_1, \dots, \varepsilon_n$ are consistent with a given distribution F_ε . If however F_ε belongs to, say, a one-parameter family $\{F_\varepsilon(\cdot, \theta), \theta \in \mathbb{R}\}$, then one needs to know the value of the parameter θ . A straightforward procedure would be to estimate it and use the statistic

$$\sup_{x \in \mathbb{R}} \left| \sum_{i=1}^n 1_{\{\hat{\varepsilon}_i < x\}} - F_\varepsilon(x; \hat{\theta}_n) \right|,$$

where $F_\varepsilon(x; \hat{\theta}_n)$ is the distribution function $F_\varepsilon(x) = F_\varepsilon(x; \theta)$ in which the parameter θ has been replaced with its estimator $\hat{\theta}_n$.

Therefore, this section is devoted to study the limiting behaviour of

$$\hat{L}_n(x) := \sum_{i=1}^n \left(1_{\{\hat{\varepsilon}_i \leq x\}} - F_\varepsilon(x; \hat{\theta}_n) \right).$$

The results below may be seen as counterpart to the asymptotic results for

$$L_n(x) := \sum_{i=1}^n \left(1_{\{\varepsilon_i \leq x\}} - F_\varepsilon(x; \theta) \right),$$

see [13] for results and references therein for more discussion on this approach.

Many estimators $\hat{\theta}_n$ of θ can be obtained with help of partial sums $\sum_{i=1}^n H(\hat{\varepsilon}_i)$, where H is a function that does not depend on n . Let us note that from Theorem 3.1 we may have two scenarios for $\alpha_\varepsilon < 1/2$:

- (A) $\sigma_{n,2}^{-1} (\sum_{i=1}^n H(\hat{\varepsilon}_i) - \mathbb{E}[H(\varepsilon_i)])$ converges in distribution to a nondegenerate random variable;
- (B) $\sigma_{n,2}^{-1} (\sum_{i=1}^n H(\hat{\varepsilon}_i) - \mathbb{E}[H(\varepsilon_i)]) = o_P(1)$.

Example 3.6. Consider $H(u) = u^2$ which yields the estimator of $\text{Var}(\varepsilon)$. We obtain for $\alpha_\varepsilon < 1/2$:

$$\sigma_{n,2}^{-1} \left(\sum_{i=1}^n H(\hat{\varepsilon}_i) - \mathbb{E}[H(\varepsilon_i)] \right) \xrightarrow{d} \int f_\varepsilon^{(1)}(v) dH(v) \left(Z_1^2 - \frac{1}{2} Z_2^2 \right) = 2 \left(Z_1^2 - \frac{1}{2} Z_2^2 \right).$$

Consider now $H(u) = u^3$. We have for $\alpha_\varepsilon < 1/2$:

$$\sigma_{n,2}^{-1} \left(\sum_{i=1}^n H(\hat{\varepsilon}_i) - \mathbb{E}[H(\varepsilon_i)] \right) \xrightarrow{d} 6 \int v f_\varepsilon(v) dv \left(Z_1^2 - \frac{1}{2} Z_2^2 \right).$$

Consequently, if f_ε is symmetric, then the right hand side is simply 0 and thus we are in scenario (B).

In what follows, we will write $f_\varepsilon(\cdot; \theta)$ to indicate the density with the true parameter θ .

Corollary 3.7. *Assume that $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n H(\hat{\varepsilon}_i)$ and $\theta = E[H(\varepsilon)]$. Under the conditions of Corollary 3.3, we have*

$$\frac{1}{\sigma_{n,2}} \hat{L}_n(x) \Rightarrow \left(f_\varepsilon^{(1)}(x; \theta) + f_\varepsilon(x; \theta) \int f_\varepsilon^{(1)}(u; \theta) dH(u) \right) \left(Z_2 - \frac{1}{2} Z_1^2 \right),$$

and

$$n^{-1/2} \hat{L}_n(x) \Rightarrow W_1(x) + f_\varepsilon(x; \theta) \int W_1(u) dH(u),$$

respectively for $\alpha_\varepsilon < 1/2$ and $\alpha_\varepsilon > 1/2$, provided that the integrals at the right hand sides are finite.

Remark 3.8. In case $\alpha_\varepsilon > 1/2$, one needs very restrictive conditions on finiteness of $\int W_1(u) dH(u)$. In principle, it requires that H has a finite support.

Remark 3.9. We note that rates of convergence for $\hat{L}_n(\cdot)$, residual empirical process with estimated parameters, are the same as for $\hat{K}_n(\cdot)$, the ordinary residual empirical process. This is different as compared to $K_n(\cdot)$ and its "estimated" version; see [13].

Proof of Corollary 3.7. We conduct the proof for $\alpha_\varepsilon < 1/2$. For a function $g(x; \theta)$ denote by $\nabla_\theta^r g(x; \theta)$ its r th order derivative with respect to θ , evaluated at $\theta = \theta$. In particular, $\nabla = \nabla^1$. Then

$$\hat{L}_n(x) = \hat{K}_n(x) + n(\theta - \hat{\theta}_n) \nabla_\theta F_\varepsilon(x; \theta) + \frac{1}{2} n(\theta - \hat{\theta}_n)^2 \nabla_\theta^2 F_\varepsilon(x; \theta_n^*)$$

with some $\hat{\theta}_n^*$ such that $|\hat{\theta}_n^* - \hat{\theta}_n| \leq |\theta - \hat{\theta}_n^*|$. Therefore

$$\begin{aligned} \hat{L}_n(x) &= \hat{K}_n(x) + f_\varepsilon(x; \theta) \left(\sum_{i=1}^n (E[H(\varepsilon)] - H(\hat{\varepsilon}_i)) \right) + o_P(\sigma_{n,2}) \\ &= \hat{K}_n(x) - f_\varepsilon(x; \theta) \left(\int H(u) d\hat{K}_n(u) \right) + o_P(\sigma_{n,2}) \\ &= \hat{K}_n(x) + f_\varepsilon(x; \theta) \left(\int \hat{K}_n(u) dH(u) \right) + o_P(\sigma_{n,2}) \end{aligned}$$

and the result follows from Corollary 3.3.

3.5 Nonparametric regression

Now, we establish the result for nonparametric regression case. It is assumed that $m(\cdot)$ is estimated by the usual Nadaraya-Watson estimator, i.e.

$$\hat{m}(x) = \hat{m}_b(x) = \frac{1}{n b \hat{f}_b(x)} \sum_{j=1}^n Y_j K_b(x - X_j), \quad (19)$$

with

$$\hat{f}_b(x) = \frac{1}{nb} \sum_{j=1}^n K_b(x - X_j), \quad (20)$$

where $K_b(\cdot) = K(\cdot/b)$ and $K(\cdot)$ is a positive kernel, which fulfills standard conditions: $\int K(u) du = 1$, $\int uK(u) du = 0$ and $\int u^2K(u) du < \infty$.

Here we shall assume for simplicity that

(P1) Predictors are i.i.d.

Results can be extended to LRD stationary predictors using estimates from [15].

Corollary 3.10. *Assume (P1) and (E). Assume that $F_\eta(\cdot)$ is 5 times differentiable with bounded, continuous and integrable derivatives. Also, suppose that the bandwidth fulfills*

$$b + (nb)^{-1} \rightarrow 0, \quad (21)$$

$$b\sigma_{n,1}^2/n \rightarrow \infty. \quad (22)$$

(a) If $\alpha_\varepsilon < 1/2$, and

$$b^2n/\sigma_{n,1} + nb^4/\sigma_{n,2} + b^2\sigma_{n,1}/\sigma_{n,2} + \frac{\sigma_{n,1}^2}{\sigma_{n,2}^2nb} \rightarrow 0. \quad (23)$$

then

$$\frac{1}{\sigma_{n,2}} \hat{K}_n(x) \Rightarrow f_\varepsilon^{(1)}(x)(Z_2 - \frac{1}{2}Z_1^2),$$

where Z_1, Z_2 are defined in (4).

(b) If $\alpha_\varepsilon > 1/2$, and

$$b^2n/\sigma_{n,1} + nb^4/\sqrt{n} + b^2\sigma_{n,1}/\sqrt{n} + \sigma_{n,1}^2/(n^2b) \rightarrow 0. \quad (24)$$

then $n^{-1/2}\hat{K}_n(x) \Rightarrow W_1(x)$.

Remark 3.11. The condition (21) is standard in nonparametric estimation. With the standard bandwidth choice $b = Cn^{-1/5}$ (see e.g. [14]) condition (22) is valid for $\alpha_\varepsilon < 4/5$. Likewise, one can easily verify that (23) holds for $\alpha_\varepsilon < 4/5$ as well and so for all $\alpha_\varepsilon < 1/2$. Finally, (24) holds for $1/5 < \alpha_\varepsilon < 4/5$ and so for all $1/2 < \alpha_\varepsilon < 4/5$.

Proof of Corollary 3.10. In the nonparametric regression model we have

$$\Delta_i = \hat{m}_b(X_i) - m(X_i) = R_b(X_i) + \frac{1}{nb\hat{f}_b(X_i)} \sum_{j=1}^n K_b(X_i - X_j)\varepsilon_j, \quad (25)$$

where

$$R_b(y) = \frac{1}{nb\hat{f}_b(y)} \sum_{j=1}^n (m(X_j) - m(y))K_b(y - X_j). \quad (26)$$

Denote $\rho(y) = (mf)''(y) - m(y)f''(y)$. Uniformly over $\{y : f(y) > 0\}$,

$$R_b(y) - \frac{b^2 \kappa_2}{2} \frac{\rho(y)}{f(y)} = O(b^4(1 + o_P(1))). \quad (27)$$

Now, in the second part of (25), we may replace $\hat{f}_b(X_i)$ with $f(X_i)$. This is allowed since, first, $\hat{f}_b(\cdot)$ is the consistent estimator of $f(\cdot)$; second, since $K(\cdot)$ has bounded support \mathcal{I} and $f(x) > 0$, $x \in \mathcal{I}$. Define for $j \neq i$,

$$L_b(X_i, X_j) = \frac{1}{bf(X_i)} K_b(X_i - X_j).$$

We may write (recall that $L_b^*(X_i, X_j)$ is the centered version of $L_b(X_i, X_j)$)

$$\Delta_i = R_b(X_i) + E[L_b(X_1, X_2)]\bar{\varepsilon} + \frac{1}{n} \sum_{j=1}^n L_b^*(X_i, X_j)\varepsilon_j.$$

Using (27) and (52) below we argue that

$$\Delta_i = O_P(b^2) + \bar{\varepsilon} + o_P(\sigma_{n,1}/n), \quad (28)$$

uniformly in i , provided that (22) holds. Therefore, the conditions of Theorem 3.1 are fulfilled with $\Delta_0 = \bar{\varepsilon}$, $\delta_n = \sigma_{n,1}/n$ and $V = Z_1$, as long as (22) and the first part of (23) hold.

From (28),

$$\sum_{i=1}^n \Delta_i^2 = n\bar{\varepsilon}^2 + O_P(b^2\sigma_{n,1}) + O_P(nb^4) + o_P(\sigma_{n,1}^2/n) = n\bar{\varepsilon}^2 + o_P(\sigma_{n,2}),$$

if $\alpha_\varepsilon < 1/2$ and (23) holds. Likewise, if (24) holds and $\alpha_\varepsilon > 1/2$,

$$\sum_{i=1}^n \Delta_i^2 = n\bar{\varepsilon}^2 + o_P(\sqrt{n}) = O_P(\sigma_{n,1}^2/n) + o_P(\sqrt{n}) = o_P(\sqrt{n}).$$

Also, from Section 5.5 we obtain

$$\sum_{i=1}^n \Delta_i = \sum_{i=1}^n \varepsilon_i + o_P(\sigma_{n,2} \vee \sqrt{n}). \quad (29)$$

This finishes the proof. \square

3.6 Conjecture on error density estimation

We consider again the parametric regression model $Y_i = \beta_0 + \beta_1 X_1 + \varepsilon_i$. Our goal is to estimate the error density f_ε . We use the standard Parzen-Rosenblatt estimator

$$\hat{f}_{h,\Delta}(x) = \frac{1}{nh} \sum_{i=1}^n K_h(x - \hat{\varepsilon}_i). \quad (30)$$

Conjecture 3.12. Assume (P) and (E) and that (19) holds. Furthermore, assume that $F_\eta(\cdot)$ is 5 times differentiable with bounded, continuous and integrable derivatives. If $\alpha_\varepsilon < 1/2$ and

$$nh^5 \rightarrow 0, \quad \sigma_{n,2}h \rightarrow \infty. \quad (31)$$

then

$$\frac{n}{\sigma_{n,2}} \left(\hat{f}_{h,\Delta}(x) - f(x) \right) \xrightarrow{d} f_\varepsilon^{(2)}(x) \left(Z_2 - \frac{1}{2} Z_1^2 \right).$$

Remark 3.13. The first part of (31) is the standard condition which assures that a bias is negligible. As for the second part, note that

$$\frac{n}{\sigma_{n,1}} \left(\hat{f}_h(x) - f(x) \right) \xrightarrow{d} f_\varepsilon^{(1)}(x) Z_1,$$

where \hat{f}_h is the Parzen-Rosenblatt estimator of f_ε based on $\varepsilon_1, \dots, \varepsilon_n$. The above result is valid if $\sigma_{n,1}h \rightarrow \infty$. In other words, $\sigma_{n,2}h \rightarrow \infty$ is a large bandwidth condition which assures that the estimator has LRD-type behaviour. Otherwise, if $\sigma_{n,2}h \rightarrow 0$ it should be expected that the rate of convergence is \sqrt{nh} . However, the methods of this paper are not applicable to such situation. The same applies to the case $\alpha_\varepsilon > 1/2$.

The reader is referred to [20] and [15, Section 3.2] for precise results on kernel density estimation under long memory.

”Proof”. Clearly

$$n \left(\hat{f}_{h,\Delta}(x) - \mathbb{E}[f_\varepsilon(x)] \right) = \frac{1}{h} \int K \left(\frac{x-v}{h} \right) d\hat{K}_n(x-vh).$$

Using (18), and integrating by parts we write the left hand side as

$$\left(\varepsilon_{n,2} - \frac{1}{2} n \bar{\varepsilon}^2 \right) \int K(v) f_\varepsilon^{(2)}(x-vh) dv + \frac{1}{nh} \int \left(\hat{K}_n(x-vh) - S_n(x-vh) \right) dK(v).$$

Therefore, we expect

$$\frac{1}{\sigma_{n,2}} n \left(\hat{f}_{h,\Delta}(x) - \mathbb{E}[f_\varepsilon(x)] \right) \xrightarrow{d} f_\varepsilon^{(1)}(x) \left(Z_2 - \frac{1}{2} Z_1^2 \right).$$

This, however, requires a more precise o_P bound in (18). \square

4 Simulation studies

We conducted simulations justifying our results on asymptotic behaviour of supremum of the empirical process of residuals $\hat{K}_n(\cdot)$. First, we simulated $n = 100$ i.i.d. random variables ε_i , $i = 1, \dots, n$ from $N(0, 1)$ distribution. Then, supremum $\sup_{x \in \mathbb{R}} K_n(x)$ was calculated. This procedure was repeated 100 times. Quartiles and standard deviation of the empirical distribution of the

supremum was calculated. Next, for the same errors, model $Y_i = 1 + 4X_i + \varepsilon_i$ was considered, and residuals were calculated using estimators of β_0, β_1 given in (14). Also, for the same errors, we assumed that $\beta_0 = 1$ is known. The same procedure was repeated with errors following LRD Gaussian process with $\alpha_\varepsilon \in \{0.2, 0.4, 0.6, 0.8\}$. The results are given in Table 1.

- Column 3: For the empirical process K_n based on errors, the variability of the supremum increases with the dependence, which is in agreement with the asymptotic theory for the LRD-based empirical processes.
- Column 4: We consider the empirical process L_n , where $F_\varepsilon(\cdot)$ is replaced with $F_\varepsilon(\cdot; \hat{\theta}_n)$, $\hat{\theta}_n$ being sample standard deviation based on errors $\varepsilon_1, \dots, \varepsilon_n$. The results are similar to column 3. In other words, estimation of variance does not influence asymptotic behaviour of the empirical process. This agrees with theoretical results; see [13, Remark 1.6]. This happens since variance can be estimated with rate $\sigma_{n,2} \vee \sqrt{n}$, whereas the rate of convergence for $K_n(\cdot)$ is $\sigma_{n,1}$.
- Column 5: We consider the residual-based empirical process \hat{K}_n in the linear regression model. Both slope and intercept are estimated. We note that the variability for $\alpha_\varepsilon = 0.8$ or $\alpha_\varepsilon = 0.6$ is almost the same as for i.i.d. case. In other words, LRD does not play any role, which is in agreement with Corollary 3.3.
- Column 6: Results for the residual-based empirical process \hat{L}_n with estimated variance are similar as for \hat{K}_n . Recall that Corollary 3.7 indicates that rates of convergence for \hat{L}_n is the same as for \hat{K}_n .
- Column 7: We consider \hat{K}_n , but the intercept is assumed to be known. Results are similar to Column 3. In other words, in case of known intercept the asymptotic behaviour of \hat{K}_n is similar to K_n ; see Remark 3.4.

5 Technical details

Let $\mathcal{H}_i = \sigma(\eta_i, \eta_{i-1}, \dots)$. Let $\mathbf{u} = (u_1, \dots, u_n)$ be a vector of scalars. Define

$$Z_n(x; \mathbf{u}) = \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq x+u_i\}} - F_\varepsilon(x+u_i)) - \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq x\}} - F_\varepsilon(x)).$$

The process $Z_n(x; \mathbf{u})$ is written as

$$\begin{aligned} Z_n(x; \mathbf{u}) &= \sum_{i=1}^n (\mathbf{1}_{\{x < \varepsilon_i \leq x+u_i\}} - \mathbb{E} [\mathbf{1}_{\{x < \varepsilon_i \leq x+u_i\}} | \mathcal{H}_{i-1}]) \\ &+ \sum_{i=1}^n (\mathbb{E} [\mathbf{1}_{\{x < \varepsilon_i \leq x+u_i\}} | \mathcal{H}_{i-1}] - \mathbb{E} [\mathbf{1}_{\{x < \varepsilon_i \leq x+u_i\}}]) =: M_n(x; \mathbf{u}) + N_n(x; \mathbf{u}). \end{aligned} \tag{32}$$

		K_n	L_n	\hat{K}_n	\hat{L}_n	$\hat{K}_n; \beta_0 = 1$	$\hat{L}_n; \beta_0 = 1$
i.i.d.	Q_1	0.0416	0.0392	0.0467	0.0413	0.0419	0.0376
	Q_3	0.0880	0.0859	0.0656	0.0592	0.0873	0.0789
	s	0.0314	0.0315	0.0169	0.0146	0.0312	0.0313
$\alpha_\varepsilon = 0.8$	Q_1	0.0307	0.0274	0.0473	0.0448	0.0346	0.0278
	Q_3	0.0994	0.0940	0.0686	0.0637	0.0963	0.0908
	s	0.0484	0.0494	0.0149	0.013	0.0494	0.0504
$\alpha_\varepsilon = 0.6$	Q_1	0.0303	0.0150	0.0488	0.0447	0.0274	0.0147
	Q_3	0.1285	0.1237	0.0718	0.0646	0.1303	0.1192
	s	0.0758	0.0786	0.0151	0.0139	0.0772	0.0797
$\alpha_\varepsilon = 0.4$	Q_1	0.0062	0.0038	0.0504	0.0471	0.0072	0.0041
	Q_3	0.1471	0.1353	0.0784	0.0662	0.1479	0.1353
	s	0.0858	0.0852	0.0194	0.0147	0.0850	0.0845
$\alpha_\varepsilon = 0.2$	Q_1	0.0015	0.0023	0.0535	0.0418	0.0021	0.0017
	Q_3	0.2870	0.2714	0.0826	0.0645	0.2978	0.2770
	s	0.1911	0.1851	0.0218	0.0178	0.1906	0.1850

Table 1: Simulated values of different dispersion measures.

Recall now that $\Delta = (\Delta_1, \dots, \Delta_n)$. Recalling (9), we decompose

$$\begin{aligned}
& \hat{K}_n(x) - K_n(x) = \\
&= M_n(x; \Delta) + N_n(x; \Delta) + f_\varepsilon(x) \sum_{i=1}^n \Delta_i + \frac{1}{2} f_\varepsilon^{(1)}(x) \sum_{i=1}^n \Delta_i^2 + O\left(\sum_{i=1}^n \Delta_i^3\right).
\end{aligned} \tag{33}$$

First, in Corollary 5.2 we will establish an asymptotic expansion for the LRD part $N_n(x; \Delta)$. This will be done by considering a special structure of $N_n(x; \mathbf{u})$ (see Lemma 5.1 and (35) below) and then "replacing" \mathbf{u} with Δ under proper assumptions for the latter.

Furthermore, we have to bound $M_n(x; \Delta)$. This will be done by obtaining an uniform bound on $M_n(x; \mathbf{u})$. In this way, we may utilize the martingale structure of the latter. Clearly, $M_n(x; \Delta)$ is not a martingale. The bounds are given in Lemma 5.3 and Lemma 5.5.

5.1 LRD part

Denote $\mathbf{u}_0 = u_0 \mathbf{1}$, where $\mathbf{1}$ is the vector of dimension n , consisting of '1'. Recall that $\xi_i = \varepsilon_i - \eta_i$ and $\xi_{n,r}$ is defined in the analogous way as $\varepsilon_{n,r}$.

In the first lemma we deal with $N_n(x; \mathbf{u}_0)$. The proof is included in Section 5.1.1.

Lemma 5.1. *Assume that $F_\eta(\cdot)$ is 3 times differentiable with bounded, contin-*

uous and integrable derivatives. Then with some $0 < \nu < 1/2$ and $\delta_n \rightarrow 0$,

$$\sup_{|u_0| \leq \delta_n^{1-\nu}} \sup_{x \in \mathbb{R}} \left| N_n(x; \mathbf{u}_0) + f_\varepsilon^{(1)}(x) u_0 \xi_{n,1} \right| = O_P \left(\delta_n^{1-\nu} (\sigma_{n,2} \vee \sqrt{n}) + \delta_n^{2(1-\nu)} \sigma_{n,1} \right). \quad (34)$$

Note now that the part $N_n(x, \mathbf{u})$ in (32) can be written as

$$N_n(x; \mathbf{u}) = \sum_{i=1}^n (F_\eta(x + u_i - \xi_i) - F_\eta(x - \xi_i) - \mathbb{E}F_\eta(x + u_i - \xi_i) + \mathbb{E}F_\eta(x - \xi_i)).$$

Let us choose $\mathbf{u} = \mathbf{u}_0 + (u_{01}, \dots, u_{0n})$. If $\max_i(|u_{0i}|) = o(\delta_n)$, then applying first order Taylor expansion, and noting that ξ_i , $i \geq 1$, is LRD moving average with the same properties as ε_i , $i \geq 1$,

$$N_n(x; \mathbf{u}) - N_n(x; \mathbf{u}_0) = o(\delta_n) \sum_{i=1}^n (f_\eta(x + u_0 - \xi_i) - \mathbb{E}f_\eta(x + u_0 - \xi_i)) = o_P(\delta_n \sigma_{n,1}),$$

uniformly in u, u_0 and x , since $f_\eta^{(1)}$ is bounded and integrable. Combining this with (34), we have (recall $\nu < 1/2$)

$$\sup_{\mathbf{u}} \sup_{x \in \mathbb{R}} \left| N_n(x; \mathbf{u}) + f_\varepsilon^{(1)}(x) u_0 \xi_{n,1} \right| = O_P(\delta_n^{1-\nu} (\sigma_{n,2} \vee \sqrt{n})) + o_P(\delta_n \sigma_{n,1}), \quad (35)$$

where $\sup_{\mathbf{u}}$ is taken over all \mathbf{u} such that

$$\mathbf{u} = \mathbf{u}_0 + (u_{01}, \dots, u_{0n}), \quad \max_i(|u_{0i}|) = o(\delta_n), \quad |u_0| = O(\delta_n^{1-\nu}).$$

In this way we end up with the following corollary.

Corollary 5.2. *Assume that $F_\eta(\cdot)$ is 3 times differentiable with bounded, continuous and integrable derivatives. Assume that Δ can be written as $\Delta_0 \mathbf{1} + (\Delta_{01}, \dots, \Delta_{0n})$, where*

$$\Delta_0 = o_P(\delta_n^{1-\nu}), \quad \max_i \Delta_{0i} = o_P(\delta_n).$$

Then

$$\sup_{x \in \mathbb{R}} \left| N_n(x; \Delta) + f_\varepsilon^{(1)}(x) \Delta_0 \xi_{n,1} \right| = O_P(\delta_n^{1-\nu} (\sigma_{n,2} \vee \sqrt{n})) + o_P(\delta_n \sigma_{n,1}). \quad (36)$$

Noting that for $\alpha_\varepsilon < 1/2$ we have $\xi_{n,1} - \varepsilon_{n,1} = o_P(\sigma_{n,2})$, we may replace $\xi_{n,1}$ with $\varepsilon_{n,1}$ in the statement of Theorem 3.1.

5.1.1 Proof of Lemma 5.1

Let $F_{n,\xi}(\cdot)$ be an empirical distribution function, associated with ξ_1, \dots, ξ_n and let $F_\xi(\cdot)$, $f_\xi(\cdot)$ be, respectively, distribution and density function of any of ξ_i .

Note that ξ_i and η_i are independent for each fixed i , and $f_\xi * f_\eta = f_\varepsilon$. Recall that $\xi_{n,r}$ is defined in the analogous way as $\varepsilon_{n,r}$; see (2). From (3) we obtain that $\xi_{n,1} = O_P(\sigma_{n,1})$.

Furthermore, let

$$\tilde{S}_{n,p}(x) = \sum_{i=1}^n (\mathbf{1}_{\{\xi_i \leq x\}} - F_\xi(x)) + \sum_{r=1}^p (-1)^{r-1} F_\xi^{(r)}(x) \xi_{n,r}.$$

Note that $\tilde{S}_{n,p}$ is defined in the same way as $S_{n,p}$, but we use ξ_i 's in the former instead of ε_i 's in the latter. Nevertheless, we conclude from (5) and (6) that for $\alpha_\varepsilon < 1/2$,

$$\sigma_{n,2}^{-1} \tilde{S}_{n,1}(x) \Rightarrow f_\xi^{(1)}(x) Z_2, \quad (37)$$

where Z_2 is the same random variable as in (4). Otherwise, if $\alpha_\varepsilon > 1/2$, then

$$n^{-1/2} \tilde{S}_{n,1}(x) \Rightarrow \Psi(x), \quad (38)$$

where Ψ is a Gaussian process and the convergence is in the Skorokhod topology.

We compute

$$\begin{aligned} N_n(x; \mathbf{u}_0) &= n \int (F_\eta(x + u_0 - v) - F_\eta(x - v)) d(F_{n,\xi}(v) - F_\xi(v)) \\ &= n \int (F_{n,\xi}(v) - F_\xi(v)) (f_\eta(x + u_0 - v) - f_\eta(x - v)) dv \\ &= n \int (F_{n,\xi}(v) - F_\xi(v) + f_\xi(v) \xi_{n,1}/n) (f_\eta(x + u_0 - v) - f_\eta(x - v)) dv \\ &\quad - (f_\varepsilon(x + u_0) - f_\varepsilon(x)) \xi_{n,1} \\ &= \int \tilde{S}_{n,1}(v) (f_\eta(x + u_0 - v) - f_\eta(x - v)) dv - f_\varepsilon^{(1)}(x) u_0 \xi_{n,1} + O(u_0^2) \xi_{n,1} \\ &= \int \tilde{S}_{n,1}(v) f_\eta^{(1)}(x - v) u_0(v) dv - f_\varepsilon^{(1)}(x) u_0 \xi_{n,1} + O(u_0^2) \xi_{n,1}, \end{aligned}$$

where $u_0(v)$ lies between $x - v$ and $x + u_0 - v$. From (5) and (6) we conclude that $\sup_v |\tilde{S}_{n,1}(v)| = O_P(\sigma_{n,2} \vee \sqrt{n})$. Therefore, with a $1 > \nu > 0$,

$$\sup_{|u_0| \leq \delta_n^{1-\nu}} \sup_x \left| N_n(x; \mathbf{u}_0) + f_\varepsilon^{(1)}(x) u_0 \xi_{n,1} \right| = O_P \left(\delta_n^{1-\nu} (\sigma_{n,2} \vee \sqrt{n}) + \delta_n^{2(1-\nu)} \sigma_{n,1} \right).$$

□

5.2 Martingale part

The proofs for martingale part are standard, in particular, they are similar as in [4]. However, some details are different, since the main theorems involve non-standard scalings $n^{-1/2}$ and $\sigma_{n,2}^{-1}$, rather than $\sigma_{n,1}^{-1}$.

Lemma 5.3. *Assume that $\|f_\eta\|_\infty < \infty$.*

(a) Let $x_r = r \frac{1}{\sigma_{n,2}}$. If $\alpha_\varepsilon < 1/2$ and (8) holds, then

$$\sup_{\mathbf{u}} \max_{r \in \mathbb{Z}} |M_n(x_r; \mathbf{u})| = o_P(\sigma_{n,2}).$$

(b) Let $x_r = r \frac{\epsilon}{\sqrt{n}}$ with $\epsilon > 0$. If $\alpha_\varepsilon > 1/2$ and (11) holds, then

$$\sup_{\mathbf{u}} \max_{r \in \mathbb{Z}} |M_n(x_r; \mathbf{u})| = o_P(\sqrt{n}).$$

In both cases $\sup_{\mathbf{u}}$ is taken over all \mathbf{u} such that

$$\mathbf{u} = \mathbf{u}_0 + (u_{01}, \dots, u_{0n}), \quad \max_i (|u_{0i}|) = o(\delta_n), \quad |u_0| = O(\delta_n^{1-\nu}). \quad (39)$$

Let

$$A_n(x; y) = \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq y\}} - F_\varepsilon(y) - (\mathbf{1}_{\{\varepsilon_i \leq x\}} - F_\varepsilon(x))). \quad (40)$$

The next lemma establishes tightness-like property of the empirical process based on ε_i , $i \geq 1$. Note, however, that it cannot be concluded directly from the tightness of $\sigma_{n,1}^{-1} K_n(\cdot)$, since the different scaling is involved.

Lemma 5.4. Assume that $\|f_\eta\|_\infty < \infty$.

- If $\alpha_\varepsilon < 1/2$, then $\sup_{|y-x| \leq \sigma_{n,2}^{-1}} |A_n(x; y)| = o_P(\sigma_{n,2})$.
- If $\alpha_\varepsilon > 1/2$, then $\sup_{|y-x| \leq \epsilon n^{-1/2}} |A_n(x; y)| = O_P(\epsilon n^{-1/2})$.

Combining Lemmas 5.3 and 5.4 we obtain the following uniform behaviour of the martingale part.

Lemma 5.5. Under the conditions of Lemma 5.3 we have

$$\sup_{\mathbf{u}} \sup_{x \in \mathbb{R}} |M_n(x; \mathbf{u})| = o_P(\sigma_{n,2}) + O_P(\epsilon \sqrt{n}).$$

As in case of Corollary 5.2 we conclude the following corollary.

Corollary 5.6. Assume that $\|f_\eta\|_\infty < \infty$. Assume that Δ can be written as $\Delta_0 \mathbf{1} + (\Delta_{01}, \dots, \Delta_{0n})$, where

$$\Delta_0 = o_P(\delta_n^{1-\nu}), \quad \max_i \Delta_{0i} = o_P(\delta_n)$$

and that (8) or (11) holds respectively for $\alpha_\varepsilon < 1/2$ or $\alpha_\varepsilon > 1/2$. Then

$$\sup_{x \in \mathbb{R}} |M_n(x; \Delta)| = o_P(\sigma_{n,2}) + O_P(\epsilon \sqrt{n}). \quad (41)$$

Proof of Lemma 5.3. We prove part (a) only. The proof of the other part is analogous. Let

$$a_{n,i}(x) = a_i(x) := \mathbf{1}_{\{x \leq \varepsilon_i \leq x+u_i\}} - \mathbb{E}[\mathbf{1}_{\{x \leq \varepsilon_i \leq x+u_i\}} | \mathcal{H}_{i-1}],$$

so that $M_n(x, \mathbf{u}) = \sum_{i=1}^n a_i(x)$. We note that $\{M_n(x, \mathbf{u}), \mathcal{H}_n\}$ is a martingale array. Thus, by the Rosenthal's inequality

$$\mathbb{E}|M_n(x, \mathbf{u})|^4 \leq C \mathbb{E} \left[\left(\sum_{i=1}^n \mathbb{E}(a_i(x)^2 | \mathcal{H}_{i-1}) \right)^2 \right] + C \sum_{i=1}^n \mathbb{E} a_i^4(x).$$

Furthermore, $|a_i(x)| \leq 1$, so that

$$\mathbb{E}|M_n(x, \mathbf{u})|^4 \leq Cn \sum_{i=1}^n \mathbb{E} \left[(\mathbb{E}(a_i^2(x) | \mathcal{H}_{i-1}))^2 \right] + C \sum_{i=1}^n \mathbb{E} a_i^2(x). \quad (42)$$

Note that

$$\mathbb{E}[a_i^2(x) | \mathcal{H}_{i-1}] \leq \mathbb{E}[\mathbf{1}_{\{\varepsilon_i \leq x+|u_i|\}} | \mathcal{H}_{i-1}] - \mathbb{E}[\mathbf{1}_{\{\varepsilon_i \leq x-|u_i|\}} | \mathcal{H}_{i-1}] =: H_i^+(x) - H_i^-(x)$$

and that for each i , $H_i^+(x)$ and $H_i^-(x)$ are nondecreasing.

Introduce a partition $\mathbb{R} = \cup_{r \in \mathbb{Z}} [x_r, x_{r+1})$. Then

$$\begin{aligned} \mathbb{E} H_i^+(x_r) &= \mathbb{E} H_i^+(x_r) \cdot \sigma_{n,2} \int_{x_r}^{x_{r+1}} 1 \, dx \leq \sigma_{n,2} \mathbb{E} \left[\int_{x_r}^{x_{r+1}} H_i^+(x) \, dx \right], \\ \mathbb{E} H_i^-(x_r) &= \mathbb{E} H_i^-(x_r) \cdot \sigma_{n,2} \int_{x_{r-1}}^{x_r} 1 \, dx \geq \sigma_{n,2} \mathbb{E} \left[\int_{x_{r-1}}^{x_r} H_i^-(x) \, dx \right]. \end{aligned}$$

Thus, for arbitrary M ,

$$\begin{aligned} \sum_{r=-M}^M \mathbb{E} [H_i^+(x_r) - H_i^-(x_r)] &\leq \sigma_{n,2} \sum_{r=-M}^M \mathbb{E} \left[\int_{x_r}^{x_{r+1}} H_i^+(x) \, dx - \int_{x_{r-1}}^{x_r} H_i^-(x) \, dx \right] \\ &= \sigma_{n,2} \mathbb{E} \left[\int_{x_{-M}}^{x_M} (H_i^+(x) - H_i^-(x)) \, dx + \int_{x_M}^{x_{M+1}} H_i^+(x) \, dx - \int_{x_{-M-1}}^{x_{-M}} H_i^-(x) \, dx \right] \\ &\leq \sigma_{n,2} \mathbb{E} \left[\int_{x_{-M}}^{x_M} (H_i^+(x) - H_i^-(x)) \, dx \right] + 2. \end{aligned}$$

Note that (recall that $\xi_i = \varepsilon_i - \eta_i$)

$$H_i^+(x) - H_i^-(x) = F_\eta(x - \xi_i + |u_i|) - F_\eta(x - \xi_i - |u_i|) = \int_{-|u_i|}^{|u_i|} f_\eta(x - \xi_i + y) \, dy, \quad (43)$$

and

$$|H_i^+(x) - H_i^-(x)| \leq 2|u_i| \sup_x f_\eta(x). \quad (44)$$

Using (43) we obtain

$$\begin{aligned}
& \sum_{r=-M}^M \mathbb{E} [H_i^+(x_r) - H_i^-(x_r)] \\
& \leq 1 + \sigma_{n,2} \mathbb{E} \left[\int_{x-M}^{x_M} \int_{-|u_i|}^{|u_i|} f_\eta(x - \xi_i + y) dy dx \right] \\
& \leq 1 + \sigma_{n,2} \mathbb{E} \left[\int_{-|u_i|}^{|u_i|} \int_{-\infty}^{\infty} f_\eta(x + \xi_i + y) dx dy \right] \\
& = 2 + \sigma_{n,2} \mathbb{E} \left[\int_{-|u_i|}^{|u_i|} 1 dy \right] = 2 + 2\sigma_{n,2}|u_i|.
\end{aligned} \tag{45}$$

Combining (43), (44) and (45),

$$\sum_{r=-M}^M \mathbb{E} \left[(H_i^+(x_r) - H_i^-(x_r))^2 \right] \leq C|u_i| + C\sigma_{n,2}u_i^2. \tag{46}$$

Also, $\mathbb{E}a_i^2(x) \leq \mathbb{E}[H_i^+(x_r) - H_i^-(x_r)]$. By Markov inequality and (42),

$$\begin{aligned}
P \left(\max_r \frac{1}{\sigma_{n,2}} |M_n(x_r, \mathbf{u})| > 1 \right) & \leq \frac{1}{\sigma_{n,2}^4} \sum_r \mathbb{E} M_n^4(x_r, \mathbf{u}) = \frac{1}{\sigma_{n,2}^4} \sum_r \mathbb{E} \left(\sum_{i=1}^n a_i(x_r) \right)^4 \\
& \leq \frac{1}{\sigma_{n,2}^4} \left\{ Cn \sum_r \sum_{i=1}^n \mathbb{E} \left[(\mathbb{E}(a_i^2(x_r) | \mathcal{H}_{i-1}))^2 \right] + C \sum_r \sum_{i=1}^n \mathbb{E} a_i^2(x_r) \right\} \\
& \leq \frac{C}{\sigma_{n,2}^4} \left\{ n \sum_{i=1}^n |u_i| + n\sigma_{n,2} \sum_{i=1}^n u_i^2 + n + \sigma_{n,2} \sum_{i=1}^n |u_i| \right\}.
\end{aligned}$$

The bound converges to 0 under the conditions (8) and (39). \square

Proof of Lemma 5.4. Similarly to (32), $A_n(x; y)$ is decomposed as $\tilde{M}_n(x; y) + \tilde{N}_n(x; y)$, where $\tilde{M}_n(x; y)$ is the martingale part and $\tilde{N}_n(x; y)$ is the LRD part. We have

$$\tilde{N}_n(x; y) = \sum_{i=1}^n (\mathbb{E}[1_{\{x < \varepsilon_i < y\}} | \mathcal{H}_{i-1}] - (F_\varepsilon(y) - F_\varepsilon(x))) \leq n \|f_\eta + f_\varepsilon\|_\infty |y - x|. \tag{47}$$

From [19, Lemma 14], $\sup_{|y-x| \leq \varepsilon n^{-1/2}} |\tilde{M}_n(x; y)| = O_P(\varepsilon n^{-1/2})$. Therefore, the case $\alpha_\varepsilon > 1/2$ is proven.

Furthermore, for $\alpha_\varepsilon < 1/2$,

$$\sup_{|y-x| \leq \sigma_{n,2}^{-1}} |\tilde{M}_n(x; y)| \leq 2 \sup_{x \in \mathbb{R}} \left| \sum_{i=1}^n (1_{\{\varepsilon_i \leq x\}} - \mathbb{E}[1_{\{\varepsilon_i \leq x\}} | \mathcal{H}_{i-1}]) \right| = o_P(\sigma_{n,2}).$$

\square

Proof of Lemma 5.5. We start with $\alpha_\varepsilon < 1/2$. We can rewrite $a_i(x)$ as follows:

$$a_i(x) = \mathbf{1}_{\{\varepsilon_i \leq x+u_i\}} - \mathbf{1}_{\{\varepsilon_i \leq x\}} - (F_\eta(x - \xi_i + u_i) - F_\eta(x - \xi_i)).$$

Let $x \in [x_r, x_{r+1})$, since $\mathbf{1}_{\{\varepsilon_i \leq x\}}$ and $F_\eta(x)$ are nondecreasing functions with respect to x we have

$$\begin{aligned} a_i(x) &\leq \mathbf{1}_{\{\varepsilon_i \leq x_{r+1}+u_i\}} - \mathbf{1}_{\{\varepsilon_i \leq x\}} - (F_\eta(x - \xi_i + u_i) - F_\eta(x_{r+1} - \xi_i)) \\ &= a_i(x_{r+1}) + \mathbf{1}_{\{\varepsilon_i \leq x_{r+1}\}} - \mathbf{1}_{\{\varepsilon_i \leq x\}} + F_\eta(x_{r+1} - \xi_i + u_i) - F_\eta(x - \xi_i + u_i). \end{aligned}$$

Thus, recalling the definition of $A_n(x; y)$ given in (40),

$$\begin{aligned} M_n(x, \mathbf{u}) &= M_n(x_r; \mathbf{u}) + \sum_{i=1}^n (\mathbf{1}_{\{\varepsilon_i \leq x_{r+1}\}} - F_\varepsilon(x_{r+1}) - (\mathbf{1}_{\{\varepsilon_i \leq x\}} - F_\varepsilon(x))) \\ &\quad + \sum_{i=1}^n (F_\eta(x_{r+1} - \xi_i + u_i) - F_\eta(x - \xi_i + u_i)) \\ &=: M_n(x_r; \mathbf{u}) + A_n(x; x_{r+1}) + B_n(x; x_{r+1}; \mathbf{u}). \end{aligned}$$

Now,

$$\begin{aligned} \sup_{\mathbf{u}} \sup_{x \in \mathbb{R}} |M_n(x; \mathbf{u})| &= \sup_{\mathbf{u}} \max_{r \in \mathbb{Z}} \sup_{x \in [x_r, x_{r+1})} |M_n(x; \mathbf{u})| \leq \sup_{\mathbf{u}} \max_r |M_n(x_r; \mathbf{u})| \\ &\quad + \sup_{|x_1 - x_2| \leq \sigma_{n,2}^{-1}} A_n(x; x_{r+1}) + \sup_{\mathbf{u}} \max_r \max_{x \in [x_r, x_{r+1})} B_n(x; x_{r+1}; \mathbf{u}). \quad (48) \end{aligned}$$

On account on Lemma 5.3, the first term in (48) is $o_P(\sigma_{n,2})$. The same holds for the second part by Lemma 5.4. For last term we consider Taylor expansion for F_η :

$$F_\eta(x_{r+1} - \xi_i + u_i) = F_\eta(x - \xi_i + u_i) + f_\eta(s)(x_{r+1} - x),$$

where $s \in [x - \xi_i + u_i, x_{r+1} - \xi_i + u_i]$. Thus, the bound on $B_n(x; x_{r+1}; \mathbf{u})$ is independent of \mathbf{u}

$$B_n(x; x_{r+1}; \mathbf{u}) = \sum_{i=1}^n f_\eta(s)(x_{r+1} - x) \leq n f_\eta(s) \frac{1}{\sigma_{n,2}} = o(\sigma_{n,2})$$

since $n/\sigma_{n,2}^2 \rightarrow 0$ for $\alpha_\varepsilon < 1/2$. Thus, the proof for $\alpha_\varepsilon < 1/2$ is finished.

If $\alpha_\varepsilon > 1/2$, then with the choice $x_r = r \frac{\varepsilon}{\sqrt{n}}$ the first part in (48) is $o_P(\sqrt{n})$ and the same holds for the second part by applying Lemma 5.4. The term $B_n(x; x_{r+1}; \mathbf{u})$ is bounded by

$$B_n(x; x_{r+1}; \mathbf{u}) = \sum_{i=1}^n f_\eta(s)(x_{r+1} - x) \leq n f_\eta(s) \frac{\varepsilon}{\sqrt{n}} = O(\varepsilon \sqrt{n}).$$

□

5.3 Proofs of Theorems 3.1 and 3.2

The result of Theorem 3.1 follows from Corollary 5.2 and uniform $o_P(\sigma_{n,2})$ negligibility of the martingale part in Lemma 5.5.

Now, let $\alpha_\varepsilon > 1/2$. Corollary 5.6 implies that for each $\eta, \theta > 0$ we may choose $\epsilon > 0$ small enough so that

$$P \left(\sup_{x \in \mathbb{R}} |n^{-1/2} M_n(x, \Delta)| > \theta \right) < 1 - \eta. \quad (49)$$

Recall (33). This combined with (36) of Corollary 5.2 and (49) yields

$$\begin{aligned} \hat{K}_n(x) &= K_n(x) + M_n(x; \Delta) \\ &\quad + f_\varepsilon(x) \sum_{i=1}^n \Delta_i - f_\varepsilon^{(1)}(x) \Delta_0 \xi_{n,1} + O \left(\sum_{i=1}^n \Delta_i^2 \right) + O_P(\delta_n^{1-\nu} \sqrt{n}) + O_P(\delta_n \sigma_{n,1}), \\ &= K_n(x) + f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i + \left(f_\varepsilon(x) \sum_{i=1}^n \Delta_i - f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i - f_\varepsilon^{(1)}(x) \Delta_0 \xi_{n,1} \right) \\ &\quad + O \left(\sum_{i=1}^n \Delta_i^2 \right) + o_P(\sqrt{n}) + O_P(\delta_n \sigma_{n,1}). \end{aligned} \quad (50)$$

Application of (6) yields

$$n^{-1/2} \left(K_n(x) + f_\varepsilon(x) \sum_{i=1}^n \varepsilon_i \right) \Rightarrow W_1(x).$$

The result of Theorem 3.2 follows.

5.4 Proof of (28)

We have

$$\mathbb{E}[L_b(X_1, X_2)] \sim 1 + \frac{O(b^2)}{2} \int s^2 K(s) ds \int f^{(2)}(v) dv = 1 + O(b^2). \quad (51)$$

Consequently, $\mathbb{E}[L_b(X_1, X_2)]\bar{\varepsilon} = \bar{\varepsilon} + O_P(b^2 \sigma_{n,1}/n) = \bar{\varepsilon} + o_P(\sigma_{n,1}/n)$.

Furthermore, since central moments are bounded by ordinary moments,

$$\begin{aligned} \text{Var} \left(\frac{1}{n} \sum_{j=1}^n L_b^*(X_i, X_j) \varepsilon_j \right) &= \\ &= \frac{O(1)}{n^2} \sum_{j=1}^n \mathbb{E}[L_b^2(X_i, X_j)] + \frac{1}{n^2} \sum_{\substack{j, j'=1 \\ j \neq j'}}^n \mathbb{E}[L_b^*(X_i, X_j) L_b^*(X_i, X_{j'})] \mathbb{E}[\varepsilon_j \varepsilon_{j'}] \end{aligned}$$

It is straightforward to verify that for different indices i, j, j' ,

$$\mathbb{E}[L_b(X_i, X_j)L_b(X_i, X_{j'})] = 1 + O(b).$$

Combining this with (51) yields

$$\mathbb{E}[L_b^*(X_i, X_j) L_b^*(X_i, X_{j'})] = o(b).$$

Consequently, if (22) holds, then uniformly in i ,

$$\text{Var} \left(\frac{1}{n} \sum_{j=1}^n L_b^*(X_i, X_j) \varepsilon_j \right) = O((nb)^{-1}) + o(b\sigma_{n,1}^2/n^2) = o(\sigma_{n,1}^2/n^2). \quad (52)$$

5.5 Proof of (29)

Recall (25) and (27). Also, recall that once (27) is evaluated, we may replace $\hat{f}_b(X_i)$ with $f(X_i)$. Therefore, we have

$$\sum_{i=1}^n \Delta_i = O_P(nb^2) + \mathbb{E}[L_b(X_1, X_2)]\varepsilon_{n,1} + \frac{1}{n} \sum_{j=1}^n \tilde{L}_b^*(X_j)\varepsilon_j, \quad (53)$$

where $\tilde{L}_b(X_j) = \sum_{i=1}^n \frac{1}{bf(X_i)} K_b(X_i - X_j)$ and $\tilde{L}_b^*(X_j)$ is its centered version. Now, the variance of the third term in (53) is

$$\begin{aligned} & \frac{1}{n^2} \sum_{j,j'=1}^n \mathbb{E}[\varepsilon_j \varepsilon_{j'}] \mathbb{E} \left[\tilde{L}_b^*(X_j) \tilde{L}_b^*(X_{j'}) \right] = I_1 + I_2 + I_3 + I_4 \\ & := \frac{1}{n^2 b^2} \sum_{\substack{j,j'=1 \\ j \neq j'}}^n \mathbb{E}[\varepsilon_j \varepsilon_{j'}] \sum_{i=1}^n \text{Cov} \left[\frac{1}{f(X_i)} K_b(X_i - X_j), \frac{1}{f(X_i)} K_b(X_i - X_{j'}) \right] \\ & \quad + \frac{1}{n^2 b^2} \sum_{\substack{j,j'=1 \\ j \neq j'}}^n \mathbb{E}[\varepsilon_j \varepsilon_{j'}] \sum_{\substack{i,i'=1 \\ i \neq i'}}^n \text{Cov} \left[\frac{1}{f(X_i)} K_b(X_i - X_j), \frac{1}{f(X_{i'})} K_b(X_{i'} - X_{j'}) \right] \\ & \quad + \frac{1}{n^2 b^2} \sum_{j=1}^n \mathbb{E}[\varepsilon_j^2] \sum_{i=1}^n \text{Cov} \left[\frac{1}{f(X_i)} K_b(X_i - X_j), \frac{1}{f(X_i)} K_b(X_i - X_j) \right] \\ & \quad + \frac{1}{n^2 b^2} \sum_{j=1}^n \mathbb{E}[\varepsilon_j^2] \sum_{\substack{i,i'=1 \\ i \neq i'}}^n \text{Cov} \left[\frac{1}{f(X_i)} K_b(X_i - X_j), \frac{1}{f(X_{i'})} K_b(X_{i'} - X_j) \right]. \end{aligned}$$

We start with I_1 . We claim that

$$I_1 = \frac{O(1)}{n^2 b^2} \sigma_{n,1}^2 \left(\underbrace{nb^2}_{i,j,j' \text{ different}} + \underbrace{b}_{j \neq j', i=j \text{ or } i=j'} \right) = O(\sigma_{n,1}^2/n).$$

Indeed, let us verify the case when i, j, j' are different. We have (recall (P1))

$$\begin{aligned}
& \text{Cov} \left[\frac{1}{f(X_i)} K_b(X_i - X_j), \frac{1}{f(X_i)} K_b(X_i - X_{j'}) \right] \\
& \leq \mathbb{E} \left[\frac{1}{f(X_i)} K_b(X_i - X_j) \frac{1}{f(X_i)} K_b(X_i - X_{j'}) \right] \\
& = \iiint \frac{1}{f(u)} K_b(u - v) \frac{1}{f(u)} K_b(u - v') f(v) f(v') du dv dv' \\
& = b^2 \iiint \frac{1}{f(u)} K(s) \frac{1}{f(u)} K(s') f(u - sb) f(u - s'b) du ds ds' = O(b^2).
\end{aligned}$$

In I_2 , the term with all indices i, i', j, j' different vanishes (recall that we work under (P1)). The other terms are verified in the similar way as for I_1 , by computing expected values of products instead of covariances. We obtain:

$$I_2 = \frac{O(1)}{n^2 b^2} \sigma_{n,1}^2 \left(\underbrace{nb}_{i \neq i', j \neq j', i=j, i' \neq j'} + \underbrace{1}_{i \neq i', j \neq j', i=j, i'=j'} \right) = O(\sigma_{n,1}^2 / (nb)).$$

Similarly,

$$I_3 = \frac{1}{n^2 b^2} \sum_{j=1}^n \mathbb{E}[\varepsilon_j^2] \left(\sum_{i=1}^n O(b) + 1 \right) = O(b^{-1}) = o(n).$$

Finally, for I_4 let us note that with i, i', j different we obtain

$$\text{Cov} \left[\frac{1}{f(X_i)} K_b(X_i - X_j), \frac{1}{f(X_{i'})} K_b(X_{i'} - X_j) \right] = 0.$$

Thus,

$$I_4 = \frac{O(1)}{n^2 b^2} n \left(\underbrace{nb}_{i \neq i', i=j \text{ or } i'=j} \right) = O(b^{-1}) = o(n).$$

From (53), (51) and the above estimates we obtain

$$\sum_{i=1}^n \Delta_i = O_P(nb^2) + \varepsilon_{n,1} + O_P(b^2 \sigma_{n,1}) + O_P\left(\frac{\sigma_{n,1}}{\sqrt{nb}}\right) + o_P(\sqrt{n}).$$

If $\alpha_\varepsilon < 1/2$ and (23) holds, then the above estimate is $o_P(\sigma_{n,2})$. Likewise, if $\alpha_\varepsilon > 1/2$ and (24) holds, then the bound is $o_P(\sqrt{n})$. Thus, (29) is proven. \square

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